

A Field Theory Model With a New Lorentz-Invariant Energy Scale

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Abstract

A framework is proposed that allows to write down field theories with a new energy scale while explicitly preserving Lorentz invariance and without spoiling the features of standard quantum field theory which allow quick calculations of scattering amplitudes. If the invariant energy is set to the Planck scale, these deformed field theories could serve to model quantum gravity phenomenology. The proposal is based on the idea, appearing for example in Deformed Special Relativity, that momentum space could be curved rather than flat. This idea is implemented by introducing a fifth dimension and imposing an extra constraint on physical field configurations in addition to the mass shell constraint. It is shown that a deformed interacting scalar field theory is unitary. Also, a deformed version of QED is argued to give scattering amplitudes that reproduce the usual ones in the leading order. Possibilities for experimental signatures are discussed, but more work on the framework's consistency and interpretation is necessary to make concrete predictions.

1 Introduction

There is a large discrepancy between the physical pictures painted by standard quantum field theory on the one hand and, on the other, candidates for fundamental theories of nature such as loop quantum gravity or string theory. One describes the world in terms of particles on a smooth background, while the others involve much stranger objects like spin-networks, strings, or membranes. Moreover, the realms of validity of standard field theories and quantum gravity theories are separated by many orders of magnitude. The situation prompts many questions as to what kind of physics governs the intermediate energy scales.

Dimensional arguments suggest that quantum gravity effects should become important at energy scales close to the Planck energy, $E_{Planck} = \sqrt{\hbar c/G} \sim 10^{22}$ MeV. For particle accelerator experiments, this is an extremely high energy. Nevertheless, it has been realized in recent years that experiments detecting ultra-high energy particles from cosmological sources may already be or may soon become sensitive to phenomena around the Planck scale [?]. It would be therefore both interesting and useful to have some models that could predict or reproduce the results of these experiments. The purpose of this paper is to formulate and study a model that could bridge the divide between standard physics and quantum gravity, and thereby serve to understand Planck scale phenomenology. The aim is to remain conceptually close to standard quantum field theory so that computations can be assigned meaningful interpretations, but to nevertheless consider some general features that are expected to arise from quantum gravity.

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A simple way to incorporate the Planck scale into a field theory is to add new terms to the standard model Lagrangian. Models constructed in this manner have been extensively studied and their free parameters have already been tightly constrained [?, ?, ?]. However, since such approaches usually break Lorentz invariance in their very formulation, they are at odds with the candidates for fundamental theories of quantum gravity such as loop quantum gravity or string theory.

A candidate for the resolution of the problem with breaking Lorentz symmetry is Deformed Special Relativity (DSR). In DSR, the momentum space transformations of special relativity are modified so that the energy (or momentum - the details are model dependent) of a particle is saturated at a particular level much like the velocity is bounded by c [?]. The new invariant energy scale of the DSR models is called κ in concordance to earlier work on quantum deformations of the Poincare algebra [?]. An important aspect of DSR is that the scale κ can be understood as setting a curvature scale in the momentum space. The curvature gives rise to one of the characteristic features of DSR that the action of boosts on momentum variables is non-linear. (Indeed, it is the non-linearity of the transformations that implements the invariance property of the scale κ .) Interest in the DSR approach is enhanced by indications that it may actually descend, in a certain limit, from candidates for a full theory of quantum gravity [?, ?].

In this paper, the core idea of DSR that momentum space could be curved rather than flat is used to deform standard quantum field theory. The goal of the work is to write a field theory that incorporates the scale κ in a Lorentz invariant way and that is at the same time relatively easy to calculate with. Although the motivation for this work comes from the DSR program, the resulting model is distinct from constructions already proposed [?, ?, ?]. In fact, the approach in this paper is designed to avoid many of the complications, such as non-linear and non-associative momentum addition rules or non-commutative geometry, that arise in the other works. The exact relation between the proposed model and the other ones based on orthodox DSR is not addressed in detail and remains to be understood.

In the next section, the idea of curved momentum space is explained in more detail. Very importantly, it is pointed out that some well-known features of special relativity can be used as a guide in formulating a theory on (anti-)de Sitter momentum space. The discussion suggests to use framework that relies on an auxiliary dimension but characterizes physical particle states by a new constraint. Effectively, the proposal is to treat the new energy scale in much the same way as the first energy scale characterizing quantum field theories, the mass, is dealt with. In section ??, the formalism is used to define a deformed ϕ^4 scalar field theory. The momentum space constraints are implemented at the level of the field expansions, interactions are defined through Feynman diagrams, and scattering amplitudes are obtained via a new set of Feynman rules. The deformed theory is shown to be unitary to second order in the coupling. In section ??, the deformation procedure is applied to quantum electrodynamics. Properties and consequences of having momentum conservation in the extra dimension are discussed. Finally, section ?? reviews the positive as well as the negative features of the proposed deformed field theories.

2 Curved Momentum Space

The structure of momentum space plays an important role in classical and quantum mechanics. It is usually assumed to be a flat, four dimensional manifold. In trying to introduce a new invariant scale into quantum field theory, it is tempting to implement it as a curvature in the particle momentum space. This idea is in fact the corner-stone of Deformed Special Relativity (DSR) [?], although it has already been explored earlier in other contexts [?]. The purpose of this section is to review the concept of curvature in momentum space. It is discussed in the context of both Deformed Special Relativity and standard Special Relativity, and the insights gained are used to propose an approach to field theory with an invariant energy scale based on an auxiliary dimension.

A brief explanation of notation: Latin indices are three-dimensional ($i, j, k = 1, 2, 3$), Greek indices are four-dimensional ($\mu, \nu = 0, 1, 2, 3$), the metric on four-dimensional flat momentum space is $\eta_{\mu\nu} = \text{diag}(+, -, -, -)$ so that $p^2 = p_0^2 - p_i^2$. Some of the discussion refers to a large auxiliary dimension labelled by coordinates p_4 . Any dependence on this extra dimension, as well as its signature, is always made explicit.

Deformed Special Relativity

Although several reviews of the DSR formalism at varying level of technical detail are available [?], it is useful to briefly summarize some of its features here. It turns out that the momentum variables of DSR parametrize a four-dimensional space of positive curvature - a deSitter space [?]. A DSR model can be constructed from this insight as follows. Four-dimensional deSitter space can be viewed as the hyperboloid in a five-dimensional space with coordinates (p_0, p_i, p_4) satisfying the constraint

$$p_0^2 - p_i^2 - p_4^2 = -\kappa^2; \quad (1)$$

the deSitter radius κ being the new invariant scale. It follows from the form of the constraint that it is invariant under an $SO(1, 3)$ symmetry whose boost generators act as follows

$$[N_i, p_0] = ip_i, \quad [N_i, p_j] = i\delta_{ij}p_0, \quad [N_i, p_4] = 0. \quad (2)$$

Since the surface (??) is four-dimensional, it is possible to describe it using a set of only four intrinsic coordinates. The map from the five embedding coordinates (p_μ, p_4) to these new four-dimensional variables is not unique and different choices of the map lead to the many versions (bases) of DSR. For example, planar coordinates \hat{p}_μ related to (p_μ, p_4) by

$$p_0 = \kappa \sinh \frac{\hat{p}_0}{\kappa} + \frac{1}{2\kappa} \hat{p}_i \hat{p}^i e^{\hat{p}_0/\kappa}, \quad p_i = \hat{p}_i e^{\hat{p}_0/\kappa}, \quad p_4 = \kappa \cosh \frac{\hat{p}_0}{\kappa} - \frac{1}{2\kappa} \hat{p}_i \hat{p}^i e^{\hat{p}_0/\kappa} \quad (3)$$

define the bicrossproduct basis. Because the map (??) is non-trivial, the boost generators (??) act non-linearly on the four-dimensional coordinates,

$$[N_i, \hat{p}_0] = i\hat{p}_i, \quad [N_i, \hat{p}_j] = i\delta_{ij} \left(\frac{\kappa}{2} (1 - e^{-2\hat{p}_0/\kappa}) + \frac{1}{2\kappa} \hat{p}_i \hat{p}^i \right) - i \frac{\hat{p}_i \hat{p}_j}{\kappa}. \quad (4)$$

A consequence of these commutators is that $\hat{p}_i \hat{p}^i$ is bounded from above by κ^2 , thereby imposing a frame-invariant momentum cutoff. Beside the algebraic structure, there is also a non-primitive co-product map $\Delta : \mathcal{P} \rightarrow \mathcal{P} \otimes \mathcal{P}$ from one copy of the momentum algebra \mathcal{P} to a tensor product. As a consequence of the non-primitive co-product, the space-time that is dual to the momenta \hat{p}_μ is non-commutative [?]. Non-commutative geometry and the Hopf-algebraic structure of the bicrossproduct basis have therefore been the starting blocks of several approaches to constructing effective field theories in the context of DSR [?, ?, ?].

Current understanding of the proposed field theories and indeed of the whole DSR framework is still incomplete. Some of the important questions to be answered are related to how DSR should be interpreted as a physical theory and what its observable consequences are [?, ?, ?]. One of the ambiguities is due to the DSR literature making reference to several inequivalent momentum variables: the discussion thus far has mentioned the p_μ of the higher dimensional representation and the \hat{p}_μ of the bicrossproduct basis, but other variables are also possible. The different choices of coordinates are often useful for specific purposes. For example, a system of coordinates [?] in which conservation laws are linear has been used to study particle kinematics [?]. As another example, the embedding coordinates of (??) has been used to define multiple particle states that do not suffer from the so-called ‘soccer ball problem’ [?]. Some of these coordinate systems have been related to each other and to the higher dimensional one [?], but there still remain questions as to which system should be used to construct field theories.

Special Relativity

Despite the ambiguities in the interpretation of DSR, the concept of a curved momentum space is arguably as old as Special Relativity [?]. In standard Special Relativity (SR), the mass-shell condition $p_0^2 - p_i^2 = +m^2$ can also be seen, similarly to (??), to implement a curvature in the physical configuration space of relativistic particles. Interestingly, the curved momentum space of special relativity can be described using intrinsic three-dimensional variables, but such a formulation leads to non-commutative geometry just like in DSR [?]. The physical manifestation of the non-commutativity, in the case of SR, lies in the non-linear addition law for particle velocities. In some sense, therefore, it is useful to think of standard special relativity as a lower-dimensional version of DSR, or, conversely, to think of DSR as a higher dimensional generalization of standard special relativity.

There are several lessons to be learnt from comparing standard special relativity to DSR. One lesson is that the curvature in momentum space has physical consequences. In SR, that physical consequence is an upper bound on particle velocity and a velocity addition rule that is compatible with that bound. Another lesson is that, for practical purposes, it is easier to work with the higher-dimensional rather than the intrinsic variables. For example, calculations of scattering thresholds are easily done using four-momentum addition rules whereas these same calculations are much trickier in terms of the variables such as velocities that must be added in a non-trivial way. On a more philosophical level, yet another lesson to be learnt from standard special relativity is that higher-dimensional variables have a physical interpretation. As a result of SR, the unified approach to space and time (energy and momenta) is now well-established.

Higher-Dimensional Approach

This work is aimed at writing field theories that incorporate an energy scale κ into their basic equations. The motivation for this work is taken from the DSR program which suggests to view four-dimensional momentum space as a constraint surface in a five-dimensional flat space. The approach taken draws on the lessons learnt from standard SR.

For the purposes of this paper, a deformed-relativistic particle is defined in terms of a vector in flat five-dimensional momentum space subject to κ -shell and m -shell constraints. The constraints are taken to be

$$\begin{aligned} p_0^2 - p_i^2 &= m^2, \\ p_0^2 - p_i^2 - \xi p_4^2 &= -\xi \kappa^2, \end{aligned} \tag{5}$$

where the parameter $\xi = \pm 1$ is introduced to bring a little more generality than usually considered in DSR theories. The choices $\xi = +1$ and $\xi = -1$ correspond to choosing the κ -shell constraint implement a de Sitter or anti-de Sitter momentum space. Since both equations are invariant under an $SO(1,3)$ symmetry, the resultant particle theory can be said to be Lorentz-invariant.

The mass shell constraint suggests to use standard terminology like ‘energy’ and ‘momentum’ to refer to p_0 and p_i , respectively. This terminology is different and should not be confused with that used in the DSR literature where these words refer to intrinsic variables such as \hat{p}_0 and \hat{p}_i of the bicrossproduct basis. The new component of the momentum vector is referred to as simply p_4 . When both constraints are satisfied, this component of the five-momentum is fixed at

$$p_4^2 = \kappa^2 + \xi m^2; \tag{6}$$

it’s magnitude is quite large if κ is assumed to be near the Planck scale. The interpretation of p_4 is not addressed in this paper, although the lessons from standard SR suggest that it may have some physical significance. It’s large magnitude, however, is expected to only have a very small effect on low-energy particle physics.

Since the new five-dimensional space of momenta is flat, the addition (conservation) rules for these momenta should be linear just as in standard special relativity. Thus, all components of the five-momentum add linearly. It is important to note that momentum conservation in higher dimensions is a key feature of the proposed framework and sets it apart from some of the previous works; it is most similar to the proposal in [?]. This feature also leads to some interesting consequences when scattering between many particles is considered in the next sections.

3 Deformed Scalar Field Theory

This section reviews some concepts in standard quantum field theory of the scalar field and then applies these concepts to include the new momentum-space constraint. A prominent feature of this section and the next one is that the discussion does not begin with the postulation of an action or Lagrangian but rather with the definition of field expansions.

Review

In standard scalar quantum field theory, a field ϕ is expressed as a superposition of modes in four-dimensional flat momentum space. Physical configurations are selected by imposing a mass-shell constraint. Thus a physical field configuration is

$$\begin{aligned}\phi(x) &= \int \frac{d^4p}{(2\pi)^4} \phi(p) e^{ip \cdot x} (2\pi) \delta^{(4)}(p^2 - m^2) \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} \phi(p) e^{ip_0 t} e^{i\mathbf{p} \cdot \mathbf{x}}.\end{aligned}\tag{7}$$

To reach the second line, the integration $\int dp_0$ is carried out against the δ -function and positive energy solutions are selected giving $p_0 = +\sqrt{p_i^2 + m^2}$. One sees that the invariant phase space integral for the field is

$$\int d\Pi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0}.\tag{8}$$

The normalization of the functions $\phi(p)$ are fixed by an inner product defined by an integration in position space over a spatial hypersurface,

$$(\phi(x), \phi(x')) = i \int d^3x [\phi^*(x) \partial_t \phi(x') - (\partial_t \phi^*(x)) \phi(x')] = \delta^{(3)}(x - x').\tag{9}$$

In momentum space, the integral is over spatial momenta, and the time derivative brings out a factor of energy on the right hand side. The resultant normalization is

$$(\phi(p), \phi(p')) = (2\pi)^3 p_0 \delta^{(3)}(\mathbf{p} - \mathbf{p}'),\tag{10}$$

which is invariant under standard Lorentz transformations. The inner product is positive-definite for solutions with $p_0 > 0$.

The second-quantized version of the field is

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2p_0}} (a_{\mathbf{p}} e^{-ip_0 t} e^{i\mathbf{p} \cdot \mathbf{x}} + a_{\mathbf{p}}^\dagger e^{ip_0 t} e^{-i\mathbf{p} \cdot \mathbf{x}}),\tag{11}$$

where $a_{\mathbf{p}}^\dagger$ and $a_{\mathbf{p}}$ are creation and annihilation operators obeying the usual algebra $[a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')$. The propagator for the field is

$$\begin{aligned}D_F(x - y) &= \langle 0 | \phi(x) \phi(y) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} e^{-ip \cdot (x - y)} \\ &= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x - y)}.\end{aligned}\tag{12}$$

The calculation leading to the second line involves a particular but standard choice of integration contour along the p_0 direction. In momentum space, the propagator is written as

$$D_F(p) = \frac{i}{p^2 - m^2 + i\epsilon}\tag{13}$$

and is very useful in computing matrix elements in perturbation theory.

Deformation

In the deformed field theory, the fundamental momentum integral is five-dimensional and physical configurations of the scalar field are selected by imposing the two constraints (??). The field expansion is defined by

$$\phi(x, x_4) = \int \frac{d^5 p}{(2\pi)^5} \phi(p, p_4) e^{ip \cdot x} e^{ip_4 x_4} (2\pi) \delta^{(4)}(p^2 - m^2) (2\pi) \delta^{(5)}(p^2 - \xi p_4^2 + \xi \kappa^2). \quad (14)$$

Integrating out first the κ -shell δ -function against $\int dp_4$ gives

$$\phi(x, x_4) = \int \frac{d^4 p}{(2\pi)^4} \frac{1}{2p_4} \phi(p, p_4) e^{ip \cdot x} e^{ip_4 x_4} (2\pi) \delta^{(4)}(p^2 - m^2), \quad (15)$$

with the p_4 momentum is now fixed at $p_4 = \sqrt{\kappa^2 + \xi(p_0^2 - p_i^2)}$. Note that resulting field expansion is similar to (??) with the four-dimensional measure replaced

$$\int d^4 p \rightarrow \int \frac{d^4 p}{p_4}. \quad (16)$$

The latter is the integration measure for (anti)de-Sitter space of fixed radius κ . Since a good integration measure should be positive and real, the factor p_4 should be positive and real as well. Consequently, the four dimensional integrals remaining in (??) appear to be restricted to the regime $\kappa^2 + \xi(p_0^2 - p_i^2) > 0$.

Integrating out the second delta-function and selecting the solutions with positive p_0 , the field finally becomes

$$\phi(x, x_4) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2p_0} \frac{1}{2p_4} \phi(p) e^{ip_0 t} e^{i\mathbf{p} \cdot \mathbf{x}} e^{ip_4 x_4}, \quad (17)$$

where $p_0 = \sqrt{p_i^2 + m^2}$. With this substitution, the p_4 momentum becomes just a constant $p_4 = \sqrt{\kappa^2 + \xi m^2}$ and since it was assumed that $m^2 \ll \kappa^2$, p_4 is real as required for the reality of the four-dimensional measure (??).

Very importantly, note that the four-dimensional integral does not appear to be restricted if the δ -functions are removed in a different order. This suggests that the extra constraint does not really have a regulatory effect in the 4d sector as may at first appear from the discussion above. In any case, the result is that the natural phase space integral for the deformed field is

$$\int d\Pi_\kappa = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2p_0} \frac{1}{2p_4}, \quad (18)$$

which is, up to a constant, the same as in undeformed field theory.

The normalization condition can be taken as

$$(\phi(p), \phi(p')) \propto (2\pi)^3 p_0 p_4 \delta^{(3)}(\mathbf{p} - \mathbf{p}'), \quad (19)$$

which is positive definite in the selected sector where both p_0 and p_4 are positive. The quantization of the field can proceed as before by setting

$$\phi(x, x_4) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2p_0} \sqrt{2p_4}} (a_{\mathbf{p}} e^{-ip_0 t} e^{i\mathbf{p} \cdot \mathbf{x}} e^{-ip_4 x_4} + a_{\mathbf{p}}^\dagger e^{ip_0 t} e^{-i\mathbf{p} \cdot \mathbf{x}} e^{ip_4 x_4}) \quad (20)$$

and imposing the same commutators $[a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')$ on the creation and annihilation operators. The propagator for the new field is

$$D_F(x_A - x_B) = \langle 0 | \phi(x) \phi(y) | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2p_0} \frac{1}{2p_4} e^{-ip \cdot (x-y)} e^{ip_4 x_4}. \quad (21)$$

This could also be written in terms of higher dimensional integrals as

$$\begin{aligned} D_F(x - y) &= \int \frac{d^4 p}{(2\pi)^4} \left(\frac{i}{p^2 - m^2 + i\epsilon} \right) \frac{1}{2p_4} e^{-ip \cdot (x-y)} e^{ip_4 x_4} \\ &= \int \frac{d^5 p}{(2\pi)^5} \left(\frac{i}{p^2 - m^2 + i\epsilon} \right) \left(\frac{i}{p_4^2 - \kappa^2 - \xi m^2 + i\epsilon} \right) e^{-ip \cdot (x-y)} e^{-ip_4 (x_4 - y_4)}. \end{aligned} \quad (22)$$

As before, these transformations involve particular choices of contour integrations, this time along the p_0 and p_4 directions.

The propagator D_F can also be written in momentum space using the last form above. For computations with Feynman rules, this propagator will be defined as

$$D_F(p) = \frac{-1}{(p^2 - m^2 + i\epsilon)} \frac{1}{(p_4^2 - \kappa^2 - \xi m^2 + i\epsilon)}. \quad (23)$$

The appearance of an additional factor in the denominator shows that the deformation introduces new poles into the propagator: In addition to the usual poles on the p_0 axis, the propagator also has new poles along the p_4 axis. The placement of the new poles on the axis of the extra dimension will become important in the discussion of the optical theorem and unitarity in interacting theories.

Note that in defining this free theory, only those fields ϕ with positive p_0 and positive p_4 were selected. In principle, another set of scalar fields could be consistently be defined using the negative p_4 solutions to the constraint. In this paper, this sector of solutions is truncated, but it may still be interesting to explore it in more detail.

Interactions

A deformed version of an interacting theory can be defined perturbatively through Feynman diagrams and scattering amplitudes $i\mathcal{M}_\kappa$ can be computed by following a set of simple rules. Given a lack of an action from which to derive these rules, it seems reasonable to guess a set of rules and later check their consistency.

Every diagram has three main parts: external lines, internal lines, and vertices. The simplest rule dealing with the external lines is to do nothing, i.e. multiply the amplitude by 1. As for the internal lines, the previous discussion suggests to write down the deformed propagator. The remaining parts of the Feynman rules deal with the vertices. To obtain an analog for ϕ^4 theory, vertices in Feynman diagrams should be four-valent. The coupling constant can in principle be represented by any variable; it is chosen here as $-i\lambda$ in order to remain close to the notation of the undeformed theory. In accordance with the definition of the fields in five-dimensional flat momentum space, the momentum conservation δ -function usually associated with vertices should also be five-dimensional. The argument of the δ -function should be a sum of all the incoming fields' momenta in 5d. In case a Feynman

diagram contains a loop, the amplitude resulting from these rules should include an overall five-dimensional integral over this undetermined momentum. As usual, the amplitude may have to be divided by an appropriate symmetry factor.

Using these Feynman rules, it is straight-forward to compute the amplitude for the first order scattering of two incoming scalar fields into two outgoing scalar fields. The amplitude is simply proportional to λ so that $|\mathcal{M}_\kappa|^2 = \lambda^2$ just like in standard ϕ^4 theory. The amplitudes in the deformed and standard theories have the same form because there are no undetermined momenta, but the situation is not significantly more complicated for diagrams that contain loops, as is shown below.

Unitarity

A prescription for writing transition amplitudes from Feynman diagrams as defined above can only be consistent if the resulting theory is unitary. This is because unitarity is the property that ensures time-reversal invariance and the conservation of probabilities. The following discussion of the optical theorem follows [?] and checks that unitarity is preserved in ϕ^4 theory to λ^2 order.

Consider a process whereby two particles with momenta p, p_4 and p', p'_4 interact to produce another two particles with momenta k, k_4 and k', k'_4 . The S -matrix is defined as $S = 1 + iT$ where T is related to the process amplitude \mathcal{M} by

$$\langle k, k_4, k', k'_4 | T | p, p_4, p', p'_4 \rangle = i\mathcal{M}(p, p_4, p', p'_4 \rightarrow k, k_4, k', k'_4) (2\pi)^5 \delta^{(5)}(p + p' - k - k'). \quad (24)$$

Note that conservation condition of the fifth component of momentum, $p_4 + p'_4 - k_4 - k'_4 = 0$, is automatically satisfied because all these components are fixed to $\sqrt{\kappa^2 + \xi m^2}$ by the on-shell conditions. A theory is unitary if the S -matrix satisfies $SS^\dagger = 1$; in terms of the T -matrix, this translates into

$$-i(T - T^\dagger) = T^\dagger T. \quad (25)$$

The unitarity condition can be checked to first non-trivial order in perturbation theory by considering the diagram in Figure ???. In the ‘center of mass’ frame, the incoming particles have momenta (p_0, p_i, p_4) and $(p_0, -p_i, p_4)$ so that the total momentum is $(2p_0, 0, 2p_4)$. The energy p_0 can be as large as desired, but p_4 is fixed by the on-shell conditions. Therefore, the matrix element \mathcal{M} for this process can be taken to be a function of the energy p_0 alone, $\mathcal{M} = \mathcal{M}(p_0)$. The requirement that \mathcal{M} be real for low energies leads by standard arguments to the conclusion that

$$\text{Disc } \mathcal{M}(p_0) = 2i \text{Im } \mathcal{M}(p_0 + i\epsilon), \quad (26)$$

relating the discontinuity in the amplitude due to the $+i\epsilon$ prescription, to the imaginary part of the amplitude. This is useful for checking unitarity because $\text{Im } \mathcal{M}(p_0 + i\epsilon)$ also appears on the left hand side of (??).

Checking the optical theorem for the loop diagram in Figure ?? involves computing the discontinuity of the amplitude and then relating that result to the right hand side of (??). The amplitude is

$$\begin{aligned} & i\mathcal{M} \delta^{(5)}(p + p' - k - k') \\ &= \frac{\lambda^2}{2} \int \frac{d^5 q}{(2\pi)^5} \int \frac{d^5 q'}{(2\pi)^5} (f_0 f_4) (2\pi)^5 \delta^{(5)}(p + p' - q - q') \delta^{(5)}(q + q' - k - k'), \end{aligned} \quad (27)$$

Feynman graph: Δ

Figure 1: Loop diagram in ϕ^4 theory.

where

$$\begin{aligned} f_0 &= \left(\frac{1}{q^2 - m^2 + i\epsilon} \right) \left(\frac{1}{q'^2 - m^2 + i\epsilon} \right), \\ f_4 &= \left(\frac{1}{q_4^2 - \kappa^2 - \xi m^2 + i\epsilon} \right) \left(\frac{1}{q_4'^2 - \kappa^2 - \xi m^2 + i\epsilon} \right). \end{aligned} \quad (28)$$

The integration is actually over one undetermined five-momentum propagating around the loop, but it is written in terms of integrals d^5q and d^5q' and some δ -function for later convenience. Functions f_0 and f_4 contain the propagators of the internal particles. A useful feature of the model is that all the dependence (except for the overall constants) on the deformation parameter κ and the extra dimensions q_4 and q_4' can be neatly put in the function f_4 .

The discontinuity in the resulting expression can be computed using the usual cutting rules algorithm [?, ?]. In the case of the diagram in Figure ??, the cuts should be made along the two internal lines. The propagators for each of the lines consist of the factors in the function f_0 and f_4 . Thus the algorithm replaces all the factors in the denominator by δ -functions as follows

$$\begin{aligned} \frac{1}{(q^2 - m^2 + i\epsilon)} &\rightarrow -2\pi i \delta(q^2 - m^2) \\ \frac{1}{(q_4^2 - \kappa^2 - \xi m^2 + i\epsilon)} &\rightarrow -2\pi i \delta(q_4^2 - \kappa^2 - \xi m^2) \end{aligned} \quad (29)$$

and similarly for the primed variables. The δ -functions allow to evaluate the integrals along the q_0, q_0', q_4, q_4' directions, giving the discontinuity as

$$\text{Disc } \mathcal{M} = \frac{\lambda^2}{2} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2q_0} \frac{1}{2q_4} \int \frac{d^3q'}{(2\pi)^3} \frac{1}{2q_0'} \frac{1}{2q_4'} (2\pi)^5 \delta^{(5)}(p + p' - q - q'). \quad (30)$$

Returning to the right hand side of (??), the product TT^\dagger can be split by inserting an identity operator. This gives

$$\begin{aligned} &\langle k \ k_4, \ k' \ k_4' | T^\dagger T | p \ p_4, \ p' \ p_4' \rangle \\ &= \sum_n \left(\prod \int \frac{d^3q}{(2\pi)^3} \frac{1}{2q_0} \frac{1}{2q_4} \right) \langle k \ k_4, \ k' \ k_4' | T^\dagger | \{q \ q_4\} \rangle \langle \{q \ q_4\} | T | p \ p_4, \ p' \ p_4' \rangle, \end{aligned} \quad (31)$$

where the sum is over all possible intermediate configurations and the factors in parenthesis are the one-particle phase space integrals from (??) (the index characterizing the different particles is omitted in order not to confuse it with the components of the momenta, see [?] for more details on notation). For the whole expression to be of order λ^2 , each of the matrix elements $\langle \cdot | T | \cdot \rangle$ on the right hand side should correspond to just single-vertex diagrams valued λ . There is only one possible such configuration and it contains two intermediate particles. Thus (??) reduces to

$$\begin{aligned} & \langle k \, k_4, \, k' \, k'_4 | T^\dagger T | p \, p_4, \, p' \, p'_4 \rangle \\ &= \lambda^2 \int \frac{d^3 q}{(2\pi)^3} \frac{1}{2q_0} \frac{1}{2q_4} \int \frac{d^3 q'}{(2\pi)^3} \frac{1}{2q'_0} \frac{1}{2q'_4} (2\pi)^5 \delta^{(5)}(p + p' - q - q'), \end{aligned} \quad (32)$$

which is very close to the result (??). The usual discrepancy of 1/2 is due to the fact (??) should be symmetrized with respect to the intermediate particles labelled by q, q_4 and q', q'_4 .

The result shows that the proposed deformed Feynman rules generate amplitudes that satisfy the optical theorem, at least to second order in the coupling constant λ . The deformed theory is therefore unitary and consistent.

Comments

This section ends with some comments regarding the role of the fifth component of momentum in the deformed ϕ^4 theory.

Since every external particle must have its extra momentum component set to $p_4 = \sqrt{\kappa^2 + \xi m^2}$, the overall momentum conservation δ -function for any two-particle to two-particle process is trivially satisfied for this component. Moreover, this rigidity in the p_4 component implies that any process that takes two incoming particles to four or more outgoing particles is kinematically forbidden. In standard ϕ^4 theory, such processes are allowed, so the deformed ϕ^4 theory presented in this section makes some very different predictions from the standard theory at higher orders in the perturbation expansion.

An interesting question to ask at this point is whether Feynman diagram calculations at higher orders make sense in the deformed theory. In other words, is the deformed theory renormalizable? To answer this question properly, one should first understand the physical meaning of the fifth dimension in momentum space. Some aspects of this question are raised again in the next section, after having introduced a deformed version of quantum electrodynamics.

4 Deformed Electrodynamics

In this section, the deformation prescription is applied to quantum electrodynamics. A set of deformed Feynman rules for interacting fermions and photons are given that preserve diagrammatic Ward identities even in the presence of the new scale. A matrix element for a phenomenologically interesting scattering process is calculated.

Properties of the Fifth Momentum Component

Conservation of momentum in the fifth, extra, dimension can potentially have a great effect on kinematics. The primary goal of a deformed QED theory is to reproduce all the amplitudes of standard quantum electrodynamics at low energies (and, hopefully, to generate some new effects). Therefore, conservation of the momentum in the fifth-component should be compatible with all the standard QED processes - this condition can provide some interesting information about the properties of the new component of momentum, despite it not having a satisfying physical interpretation at the moment.

In the section on scalar fields, p_4 was always positive and was set to a constant $\sqrt{\kappa^2 + \xi m^2}$ for on-shell particles. When deforming quantum electrodynamics, a theory with many types of particles, several questions arise about the properties of the p_4 momentum. Should all particle types have the same magnitude of the deformation constant κ , or should the deformation scale κ depend on the particle type? Should physical particles all have their p_4 momenta carry the same sign, or should some have $p_4 > 0$ while others have $p_4 < 0$? To answer such questions, it is useful to consider some simple examples of test scattering experiments in order to extract the properties the fifth component of the momentum that are consistent with observations.

As a first test experiment, consider the reaction $e^+e^- \rightarrow \tau^+\tau^-$. In the center of mass frame, the incoming particles both have p_4 of a magnitude $\sqrt{\kappa_e^2 + \xi m_e^2}$; the constant κ_e has an e subscript to emphasize that it is labelling the κ -shell of an electron. Suppose further that the outgoing particles both have their fifth components of momentum of magnitude $\sqrt{\kappa_\tau^2 + \xi m_\tau^2}$. If momentum is conserved, $p_4^{e^+} + p_4^{e^-} = p_4^{\tau^+} + p_4^{\tau^-}$, then one can infer that if all the momenta are of the same sign, the values of κ_e and κ_τ must be different. If, on the other hand, the momenta for the electron/positron and taon/anti-taon pairs have opposite signs, then the conservation equation is satisfied trivially.

As another test example, consider a process $e^+e^- \rightarrow \tau^+\tau^-\tau^+\tau^-$ or any similar one in which there are more than two outgoing particles. Such processes cannot occur if all the p_4 have the same sign. However, momentum conservation in the fifth component is again trivially satisfied if particles and anti-particles carry p_4 of opposite signs. It should be concluded, therefore, that p_4 must be different for particles and anti-particles. No conclusion can be reached on the values of κ for the different particle species. In the work that follows, however, the values for κ for all the species are set equal for simplicity.

As a final example, consider the process $e^+ \rightarrow e^+\gamma \dots \gamma$ where an electron radiates one or several photons. It should be noted that this kind of process is forbidden in standard QED in vacuo; it can occur, however, in the presence of an external electric field, for example when the electron travels in a medium such as a liquid. Assuming that the deformation scale κ_γ for the photon is non-zero, one can also deduce that the radiation process is forbidden due to momentum conservation in the fifth dimension. It would be interesting to consider what happens to this process in the presence of an external electric field, but the following discussion, however, is concentrated only on in-vacuo processes for simplicity. Nonetheless, in order to distinguish the photon from the fermions, its deformation scale is non-zero and is labelled by $\kappa_\gamma = \mu$. The possibility that $\mu = 0$ is not considered here.

To summarize, the test scattering experiments considered suggest that particles and anti-particles should have the same deformation scale κ but opposite signs of p_4 ; the photon can

be said to carry a positive p_4 momentum, and its deformation scale is called μ . There is no indication as to whether the values of κ for different fermion species should be the same or different, but for simplicity, the following discussion assumes that all fermions have the same κ . Given these insights, the rest of this section deals with a deformed model of QED which implements these features.

Feynman Rules

The definition of QED can be split into three parts: the free fermion field, the free electromagnetic field, and the interactions between them. To start with, the deformation of the free fields can be carried out in a similar fashion as done to the scalar field in the previous section. The field expansions for the fermion fields are taken to be

$$\begin{aligned}\psi(x, x_4) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2p_0}} \frac{1}{\sqrt{2p_4}} \sum_s (a_{\mathbf{p}}^s u^s(p) e^{-ip \cdot x} + b_{\mathbf{p}}^{s\dagger} v^s(p) e^{ip \cdot x}) e^{-ip_4 x_4}, \\ \bar{\psi}(x, x_4) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2p_0}} \frac{1}{\sqrt{2p_4}} \sum_s (b_{\mathbf{p}}^s \bar{v}^s(p) e^{-ip \cdot x} + a_{\mathbf{p}}^{s\dagger} \bar{u}^s(p) e^{ip \cdot x}) e^{ip_4 x_4}.\end{aligned}\tag{33}$$

The operators a, b , their conjugates, and the four-dimensional spinors u^s, v^s and their conjugates are the usual ones. The deformation is seen in the new oscillating functions $e^{\pm ip_4 x_4}$ and in new factors of $p_4^{-1/2}$. Note that the operators $b^{s\dagger}$ and $a^{s\dagger}$ create particles with the same four-momentum p but with opposite values of the fifth momentum component p_4 , in accordance with the insights obtained earlier. The fermion propagator can be computed as in the section on the scalar field; it should be

$$\frac{\not{p} + m}{p^2 - m^2 + i\epsilon} \rightarrow \left(\frac{\not{p} + m}{p^2 - m^2 + i\epsilon} \right) \left(\frac{1}{p_4^2 - \kappa^2 - \xi m^2 + i\epsilon} \right).\tag{34}$$

The deformation of the electromagnetic field $A(x, x_4)$ can also be written in analogy to the above result for fermions. The photon propagator should thus be

$$\frac{\eta_{\mu\nu}}{q^2 + i\epsilon} \rightarrow \left(\frac{\eta_{\mu\nu}}{q^2 + i\epsilon} \right) \left(\frac{1}{q_4^2 - \mu^2 + i\epsilon} \right).\tag{35}$$

The main difference between this deformation and the one used for the fermion propagator is that here the deformation parameter is μ instead of κ .

For the purposes of introducing an interaction between fermions and the photons, it is useful to have a Lagrangian or action formulation of the theory. From the structure of the fermion propagator, it is possible to guess the action

$$S_\psi \propto \int \frac{d^4p}{(2\pi)^4} \int \frac{dp_4}{(2\pi)} [\bar{\psi}(-p, -p_4) (p_4^2 - \kappa^2) (\not{p} - m) \psi(p, p_4)].\tag{36}$$

The dimension of this action can be made to match the standard one by introducing a new dimensionless variable $\tilde{p}_4 = p_4/\kappa$ and rewriting the action as

$$S_\psi = \int \frac{d^4p}{(2\pi)^4} \int \frac{d\tilde{p}_4}{(2\pi)} [\bar{\psi}(-p, -\tilde{p}_4) (\tilde{p}_4^2 - 1) (\not{p} - m) \psi(p, \tilde{p}_4)].\tag{37}$$

Some factors of κ are absorbed into the fields ψ and $\bar{\psi}$ in this step. The possible criticism that this action leads to a higher-derivative theory when expressed in a five dimensional position-space representation together with the usual problems associated with such theories, is postponed to the comments at the end of this section.

Similarly, one can guess an action for the photon field that could give rise to the desired propagator. In terms of a dimensionless momentum $\tilde{q}_4 = q_4/\mu$, such an action could be, in Lorentz gauge,

$$S_A = \frac{1}{2} \int \frac{d^4 q}{(2\pi)^4} \int \frac{d\tilde{q}_4}{(2\pi)} [A_\mu(-q, -\tilde{q}_4)(\tilde{q}_4^2 - 1)(q^2)A^\mu(q, \tilde{q}_4)]. \quad (38)$$

To complete the definition of deformed QED, one should define the interaction vertices and the coupling constant. The interaction term is usually obtained from the action by minimally coupling the fermion field to the photon field via a replacement of partial derivatives by covariant derivatives, $\partial_\mu \rightarrow \partial_\mu + eA_\mu$. In momentum space, this replacement roughly translates to $p_\mu \rightarrow p_\mu + eA_\mu(q)$ and a condition imposing momentum conservation at the vertex. In the deformed scenario, the interaction term is taken to be

$$\begin{aligned} S_{int} = e & \left(\int \frac{d^4 p d\tilde{p}_4}{(2\pi)^5} \right) \left(\int \frac{d^4 p' d\tilde{p}'_4}{(2\pi)^5} \right) \left(\int \frac{d^4 q d\tilde{q}_4}{(2\pi)^5} \right) \\ & \times [(\tilde{p}_4^2 - 1)^{1/2} \bar{\psi}(p, p_4)] [(\tilde{q}_4^2 - 1)^{1/2} \gamma^\mu A_\mu(q, \tilde{q}_4)] [(\tilde{p}'_4 - 1)^{1/2} \psi(p', \tilde{p}'_4)] \\ & \times (2\pi)^5 \delta^{(4)}(p + p' + q) \delta(\kappa \tilde{p}_4 + \kappa \tilde{p}'_4 + \mu \tilde{q}_4). \end{aligned} \quad (39)$$

Despite the long form, this interaction term is not difficult to understand. The first line expresses that three particles are interacting in five dimensions. The factor e in front of the integrals is the usual coupling constant used in perturbation series. The second line identifies these particles as two fermions and a photon. There are also some factors depending on the dimensionless fifth component of momentum. These factors can be understood as arising from (??) by the minimal coupling prescription. The factor associated with the A field is introduced rather ad-hoc here. It can be thought of as a modification to the minimal coupling definition, as a factor that makes all fields enter symmetrically, or it can be justified a-posteriori since it simplifies some expressions in computations of transition amplitudes. The third line of (??) imposes momentum conservation in five-dimensions. The second δ -function contains κ and μ factors because momentum conservation is imposed on the original dimensionful variables.

The remaining part of defining Feynman rules is related to labelling of external lines in a diagram. Again, the guiding factor for guessing the Feynman rules is for them to reproduce the usual amplitudes as closely as possible. For this purpose, a suggested set of rules is to give each external line the usual QED factor and further divide by a factor $(\tilde{p}_4 - 1)^{1/2}$. This factor serves to cancel a similar factor that is introduced via the vertex coupling.

Because of the feature that the dependence on momenta in the extra dimension can always be factored away from the dependence on the other momenta, all amplitudes can be expressed as products of two pieces,

$$\mathcal{M}_\kappa = \mathcal{M}_s \mathcal{M}_d. \quad (40)$$

Feynman graph: Δ

Figure 2: Electron-positron scattering.

The first factor \mathcal{M}_s is the ‘standard’ piece containing the usual propagators, coupling constants, and integrals over four-momenta. The second factor \mathcal{M}_d is the ‘deformation’ piece where factors related to the new component of momentum can be found.

Examples

As a simple example of the application of the deformed Feynman rules, consider the scattering process in Figure ???. Suppose that the incoming particles are an electron/positron pair, and that the outgoing particles are a muon/anti-muon pair.

The five momenta of the electron and positron, in the center of mass, are $p_A = (p_0, p_i, p_4)$ and $p'_A = (p_0, -p_i, -p_4)$. In this frame, the five-momentum of the photon is completely determined at $(2p_0, 0, 0)$. The resultant amplitude-squared for this process, $|\mathcal{M}_\kappa|_{avg}^2$, can be obtained using the usual spin-averaging relations. It can be written separating the usual amplitude \mathcal{M} from the corrections as follows

$$|\mathcal{M}_\kappa|_{avg}^2 = \frac{1}{4} \sum_{spins} |\mathcal{M}_\kappa|^2 = \frac{1}{4} \sum_{spins} |\mathcal{M}_s|^2 |\mathcal{M}_d|^2. \quad (41)$$

Here \mathcal{M}_s is the usual amplitude of QED, while \mathcal{M}_d is a deformation factor that depends on the extra dimension and knows about the scale κ . In \mathcal{M}_d , the deformation factor in the photon propagator is cancelled by two $(\tilde{q}_4^2 - 1)^{1/2}$ factors coming from the interaction vertices. Also, the contributions of such vertex factors to the fermions are cancelled by the external line rule. So the part of the amplitude that could depend on the scale κ is actually unity, $\mathcal{M}_d = 1$. Overall, then, the deformed amplitude-squared $|\mathcal{M}_\kappa|_{avg}^2$ agrees exactly with the amplitude of standard QED. Nonetheless, the observed cross-sections in the two theories do not necessarily have to be the same. This issue is explained in more detail in the comments at the end of this section.

More significant deviations from standard QED start to appear in diagrams that contain loops, such as the electron self-energy diagram shown in Figure ??. The amplitude for this diagram is

$$\mathcal{M}_\kappa = \mathcal{M}_s \left(\int \frac{d\tilde{q}_4}{(2\pi)} 1 \right). \quad (42)$$

The standard piece \mathcal{M}_s is divergent as usual. The deformation piece, written out explicitly in terms of an integral over the fifth momentum dimension, is clearly divergent as well.

Feynman graph: Δ

Figure 3: Electron self-energy.

Therefore, it turns out that the deformed amplitude for the diagram is more problematic than in the standard theory.

It is tempting to hope that the simple form of the new infinite factor could be renormalized away in some way. In fact, one can see that divergent factors such as in (??) appear for every loop in a diagram; thus, the divergences could be removed by introducing an additional Feynman rule with the effect of either multiplying every loop in a diagram by a normalized function of the p_4 component of the loop momentum, or putting the virtual particles in the loop on the κ -shell. These seem ad-hoc at the moment but are perhaps an interesting avenue for further investigations.

Comments

The deformation factors that appear in the amplitudes are all very simple: they are either unity or integrals of unity over an undetermined component of momentum in the extra dimension. One can understand the source of this feature by defining new derived fields $\Psi(p, p_4) = (\tilde{p}_4^2 - 1)^{1/2} \psi(p, p_4)$ for the fermions and similarly for the gauge field. In terms of these variables, the action of the deformed electrodynamics looks exactly the same as that of standard QED with the only exception that all integrals are five dimensional instead of four; these extra integrals are the sources of the new divergences. Writing the action in terms of the derived fields is also helpful in seeing why the higher-order actions (??) and (??) do not cause problems for the interacting theory: interactions are defined only in the specific sector of the theory corresponding to the well-behaved derived fields. (Indeed, these properties follow almost ‘by construction’ from the minimal coupling postulate.)

It should be stressed that despite the close resemblance of the deformed theory, expressed in terms of the new derived fields, to standard QED, these two theories are not equivalent. The main difference between them is the postulate of momentum conservation in five dimensions in the deformed version, which, since p_4 is constant for on-shell particles, effectively introduces a new ‘charge’ or quantity that has to be conserved. Recall that in the scalar field theory of the previous section, conservation of momentum in the extra component had the effect of setting to zero all amplitudes of processes taking two incoming particles to four or more outgoing particles. Similar consequences could also arise in QED. A possible place where these effects could arise is in the radiation processes $e^- \rightarrow e^- \gamma \cdots \gamma$ mentioned at the beginning of this section. Deviations from the standard results would appear in diagrams with many vertices but would not depend on the energy of the incoming particle. Such behavior provides the possibility that the deformation could be tested, measured, and even

disproved in high precision experiments.

Another reason why this deformed model is non-trivial goes back to the original motivation for introducing the extra dimension, which is to implement curvature in momentum space. It may be that the physically relevant momenta are ones that, like the bicrossproduct coordinates on de Sitter space, satisfy modified dispersion relations [?, ?, ?]. In this case, then to convert the amplitudes calculated in the deformed theory to observed cross-sections, one should replace all the five-dimensional momentum variables used in the calculations by four-dimensional coordinates on the physical space. The simple amplitude (??) would give a deformed cross section σ_κ that deviates slightly from the usual σ_s ,

$$\sigma_\kappa = \sigma_s \left(1 + c_1 \frac{\tilde{p}_0}{\kappa} + \dots \right), \quad (43)$$

where c_1 is some order-unity constant and the other terms are suppressed by higher powers of κ . Computations of scattering processes in this model would then resemble the methods used in previous work on process thresholds in DSR [?, ?]. At the moment, however, a complete and satisfying explanation for the physical meaning of the various momentum variables is missing, so it would be very useful to obtain experimental feedback on this matter.

5 Discussion

In this work, a simple prescription for deforming field theories is introduced to incorporate a new energy scale κ while explicitly preserving Lorentz invariance. The motivation for writing such deformed theories is the need to study quantum gravity phenomenology. The proposed deformed models have some features that are desirable for such studies, but they also suffer from some weaknesses and ambiguities.

One of the main positive features of the proposed deformed model is that a new scale κ is introduced into the usual field theory framework while keeping the usual ability to perturbatively compute amplitudes of scattering processes. This is achieved by introducing an auxiliary dimension together with a new constraint to momentum space. (The essential features of the deformation are almost the same whether the resulting curvature in momentum space is positive or negative.) The move to a higher dimension is argued to be analogous to the shift from space to space-time that is a result of adopting special relativity. Two important consequences of this formalism are that Lorentz symmetry is explicitly preserved and that momentum conservation (in the higher-dimensional space) is linear as usual.

Field theories are constructed as deformations of standard ϕ^4 theory and standard quantum electrodynamics. The scalar field theory is explicitly shown to be unitary to second order in the coupling parameter. The deformed version of QED is argued to preserve the usual diagrammatic Ward identities (which are usually a direct consequence of gauge-invariance) and reproduce the usual scattering amplitudes to leading order. Some possibilities for seeing discrepancies between the deformed models and the standard theories are also discussed. The most promising effect stems out of the new momentum conservation rules that can suppress some higher-than-second-order interactions. Another effect, which is however dependent on the definition of the cross section and the choice of observable momentum variables, scales

as an energy over the constant κ . The possibility of such a modification has also been found in other works on scattering in the presence of a new length scale [?].

With regards to deformed quantum electrodynamics, the requirement that process amplitudes are consistent at leading order with the usually calculated quantities points to the interesting conclusion that particles and antiparticles carry momentum of opposite signs in the extra dimension. This observation is both surprising and encouraging since it implies that some properties of the new extra dimension, which in this work is treated as a byproduct of modelling quantum gravity, can be deduced from low energy experiments.

The proposed deformation model admittedly also has some weaknesses. Conceptually, one would like to have a physical interpretation for the fifth dimension, both in momentum space and the dual position space. Such an interpretation is missing and is not attempted in this paper, albeit it does seem to be related to mass. Other weaknesses of the proposed model are related to the specific deformed field theories discussed. Both of the deformed interacting field theories are defined perturbatively through a set of Feynman rules - it would be satisfying to have a formulation of the theories in terms of an action principle. Despite some comments along these lines in the section on quantum electrodynamics, one would like to have more control over this matter, particularly in the domain of quantization of the gauge field. With regards to the scattering amplitudes, the new divergences arising in loop diagrams should be understood in more detail and the question of renormalizability should be addressed systematically. As mentioned earlier, this issue may be intimately connected with the interpretation of the new extra dimension. It should be checked that the potential effects are not in contradiction with the results of existing high-precision experiments.

A lot of work remains to be done on Planck-scale phenomenology. In the context of the deformation proposal advocated in this work, future work should address and answer the weaknesses outlined above, as well as make concrete predictions for observations in real experiments. It would also be interesting to understand how the proposed approach is related, if at all, to other works on field theory in the context of Deformed Special Relativity [?, ?, ?], other field theories in curved momentum space [?], other models with invariance scales [?], or space-time-matter theory which also uses an extra dimension [?].

Acknowledgments. I would like to thank F. Girelli, S. Hossenfelder, J. Kowalski-Glikman, S. Majid, R. Mann, F. Markopoulou, R. Myers, M. Pospelov, and L. Smolin for many useful discussions. I would also like thank members of the Department of Applied Mathematics and Theoretical Physics, University of Cambridge, for their comments on a talk based on this work.

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